AREG

In the ordinary regression model the errors are assumed to be uncorrelated. The model considered here has the form

$$
\begin{aligned}
y_t &= a + \sum_{i=1}^{p} b_i x_{ti} + u_t \qquad t = 1, \dots, n \\
u_t &= \rho u_{t-1} + \varepsilon_t
\end{aligned}
\tag{1}
$$

where ε_t is an uncorrelated random error with variance σ^2 and zero mean. The error terms u_t follow a first-order autoregressive process. The constant term a can be included or excluded as specified. In the discussion below, if a is not included, it is set to be zero and not involved in the subsequent computation.

Two computational methods—Prais-Winsten and Cochrane-Orcutt—are described here.

Cochrane-Orcutt Method

Note that model **(1)** can be rewritten in two equivalent forms as:

$$
y_t - \rho y_{t-1} = a(1-\rho) + \sum_{i=1}^{p} b_i \left(x_{ti} - \rho x_{(t-1)i} \right) + \varepsilon_t \tag{2}
$$

$$
y_t - a - \sum_{i=1}^{p} b_i x_{ti} = \rho \left(y_{t-1} - a - \sum_{i=1}^{p} b_i x_{(t-1)i} \right) + \varepsilon_t \tag{3}
$$

Defining $y_t^* = y_t - \rho y_{t-1}$ and $x_{ti}^* = x_{ti} - \rho x_{(t-1)i}$ for $t = 2, \dots, n$, equation **(2)** can be rewritten as

$$
y_t^* = a(1-\rho) + \sum_{i=1}^{p} b_i x_{ti}^* + \varepsilon_t \tag{2*}
$$

Starting with an initial value for ρ, the difference y_t^* and x_{ti}^* in equation **(2*)** are computed and OLS then applied to equation **(2*)** to estimate a and b_i. These estimates in turn can be used in equation **(3)** to update $\hat{\rho}$ and the standard error of the estimate $\hat{\rho}$.

Initial Results

An initial value for ρ can be pre-set by the user or set to be zero by default. The OLS method is used to obtain an initial estimate for a (if constant term is include) and b_i.

ANOVA

Based on the OLS results, an analysis of variance table is constructed in which the degrees of freedom for regression are p, the number of X variables in equation **(1)**, while the degrees of freedom for the residual are $n-p^*-1$ if initial $\rho \neq 0$ and are $n-p^*$ otherwise. p^* is the number of coefficients in equation **(1)**. The sums of squares, mean squares, and other statistics are computed as in the REGRESSION procedure.

Intermediate Results

At each iteration, the following statistics are calculated:

Rho

An updated value for ρ is computed as

$$\hat{\rho} = \frac{\sum_{t=2}^{n} \tilde{u}_t \tilde{u}_{t-1}}{\sum_{t=1}^{n} \tilde{u}_t^2}$$

where the residuals $\tilde{u}_t$ are obtained from equation **(1)**.

Standard Error of rho

An estimate of the standard error of $\hat{\rho}$

$$\mathrm{se}\left(\hat{\rho}\right) = \sqrt{\frac{1-\hat{\rho}^2}{n-1-p^*}}$$

where $p^* = p+1$ if there is a constant term; p otherwise.

Durbin-Watson Statistic

$$DW = \frac{\sum_{i=1}^{n-1}\left(\tilde{\varepsilon}_{i+1}-\tilde{\varepsilon}_i\right)^2}{\sum_{i=1}^{n}\tilde{\varepsilon}_i^2}$$

where

$$\tilde{\varepsilon}_1 = \sqrt{1-\hat{\rho}^2}\,\tilde{u}_1$$

$$\tilde{\varepsilon}_i = \tilde{u}_i - \hat{\rho}\tilde{u}_{i-1} \qquad i = 2, \ldots, n$$

Mean Square Error

An estimate of the variance of ε_t

$$MSE = \frac{\sum_{t=2}^{n}\left(\tilde{u}_t - \hat{\rho}\tilde{u}_{t-1}\right)^2}{n-2-p^*}$$

Final Results

Iteration terminates if either all the parameters change by less than a specified value (default 0.001) or the number of iterations exceeds the cutoff value (default 10).

The following variables are computed for each case:

FIT

Fitted responses are computed as

$$\tilde{y}_1 = \hat{y}_1$$

and

$$\tilde{y}_t = \hat{y}_t + \hat{\rho}\hat{u}_{t-1} \qquad t = 2, \ldots, n$$

in which $\hat{\rho}$ is the final estimate of ρ, and

$$\hat{y}_t = \hat{a} + \sum_{i=1}^{p} \hat{b}_i x_{ti}$$

$$\hat{u}_t = y_t - \hat{y}_t \qquad t = 1, \ldots, n$$

ERR

Residuals are computed as

$$\tilde{\varepsilon}_t = y_t - \tilde{y}_t \qquad t = 2, \ldots, n$$

$$\tilde{\varepsilon}_1 = \sqrt{1 - \hat{\rho}^2 (y_1 - \tilde{y}_1)}$$

SEP

Standard error of predicted values at time t

$$SEP_1 = \sqrt{MSE}\sqrt{\left(\frac{1}{1-\hat{\rho}^2}+\tilde{h}_1\right)}$$

and

$$SEP_t = \sqrt{MSE}\sqrt{\left(1+\tilde{h}_t\right)} \qquad t = 2, \ldots, n$$

where

$$\tilde{h}_i = \mathbf{X}_i\left(\mathbf{X}^{*\prime}\mathbf{X}^*\right)^{-1}\mathbf{X}_i'$$

in which $\mathbf{X}_i$ is the predictor vector at time i with the first component 1 if a constant term is included in equation **(2*)**. $\mathbf{X}^*$ is a $(n-1)\times p^*$ design matrix for equation **(2*)**. The first column has value of $1-\hat{\rho}$ if a constant term is included in equation **(2*)**.

LCL and UCL

95% prediction interval for the future y_k is

$$\tilde{y}_k \pm t_{n-1-p^*;0.025}\, SEP_k$$

Other Statistics

Other statistics such as Multiple *R*, *R*-Squared, Adjusted *R*-Squared, and so on, are computed. Consult the REGRESSION procedure for details.

Prais-Winsten Method

This method is a modification of the Cochrane-Orcutt method in that the first case gets explicit treatment. By adding an extra equation to **(2*)**, the model has the form of

$$\begin{aligned}
&(1-\rho)y_1 = a(1-\rho) + \sum_{i=1}^{p} b_i (1-\rho) x_{1i} + (1-\rho)u_1 \\
&y_t^* = a(1-\rho) + \sum_{i=1}^{p} b_i x_{ti}^* + \varepsilon_t \qquad \text{for } t = 2, \ldots, n
\end{aligned} \tag{4}$$

Like the Cochrane-Orcutt method, an initial value of ρ can be set by the user or a default value of zero can be used. The iterative process of estimating the parameters is performed via weighted least squares (WLS). The weights used in WLS computation are $w_1 = \left(1-\hat{\rho}^2\right) \Big/ (1-\hat{\rho})^2$ and $w_i = 1$ for $i = 2, \ldots, n$. The computation of the variance of ε_t and the variance of $\hat{\rho}$ is the same as that of the WLS in the REGRESSION procedure.

Initial Results

The WLS method is used to obtain initial parameter estimates.

ANOVA

The degrees of freedom are p for regression and $n - p^*$ for residuals.

Intermediate Results

The formulas for RHO, SE Rho, DW, and MSE are exactly the same as those in the Cochrane-Orcutt method. The degrees of freedom for residuals, however, are $n-1-p^*$.

Final Results

SEP

Standard error of predicted value at time t is computed as

$$SEP_1 = \sqrt{MSE}\sqrt{\left(\frac{1}{1-\hat{\rho}^2} + \tilde{h}_1\right)}$$

$$SEP_t = \sqrt{MSE}\sqrt{\left(1 + \tilde{h}_t\right)} \qquad t = 2, \ldots, n$$

where $\tilde{h}$ is computed as

$$\tilde{h}_i = \mathbf{X}_i \left(\mathbf{X}^{*\prime}\mathbf{X}^*\right)^{-1} \mathbf{X}_i'$$

in which $\mathbf{X}_i$ is the predictor vector at time i and $\mathbf{X}^*$ is a $n \times p^*$ design matrix for equation **(4)**. If a constant term is included in the model, the first column of $\mathbf{X}^*$ has a constant value of $1-\hat{\rho}$, the first row of $\mathbf{X}^*$ is $\sqrt{w_1}\left(x_{11}, \ldots, x_{1p}\right)$, and $p^* = p + 1$.

LCL and UCL

95% prediction interval for y_k at time k is

$$\tilde{y}_k \pm t_{n-p^*;0.025}\, SEP_k$$